Journal of Global Optimization **25:** 283–303, 2003.

Ultraspherical Integral Method for Optimal Control Problems Governed by Ordinary Differential Equations

H.M. EL-HAWARY(1), M.S. SALIM(2) and H.S. HUSSIEN(3)
(1)*Department of Mathematics, Faculty of Science, Assuit University, Assuit, Egypt. (E-mail: elhawary@aun.eun.eg);* (2) *Department of Mathematics, Faculty of Science, El-Azhar University, Assuit, Egypt;* (3) *Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt. (E-mail: s.hussien@mailcity.com)*

Abstract. In this paper an ultraspherical integral method is proposed to solve optimal control problems governed by ordinary differential equations. Ultraspherical approximation method reduced the problem to a constrained optimization problem. Penalty leap frog method is presented to solve the resulting constrained optimization problem. Error estimates for the ultraspherical approximations are derived and a technique that gives an optimal approximation of the problems is introduced. Numerical results are included to confirm the efficiency and accuracy of the method.

AMS subject classification.: 45L10, 41A10, 45A05, 65L, 49M

Key words: Spectral methods, Approximation by ultraspherical polynomials, Penalty leap frog method, Differential equations, Optimal control.

1. Introduction

Spectral methods using expansion in orthogonal polynomials such as Chebyshev or Legendre polynomials have proven successful in the numerical approximation of various boundary value problems, see for instance, Canuto et al. [1], Gottlib and Orszag [13] and Szegö [22]. If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded. This choice of trial functions is responsible for the superior approximation properties of spectral methods when compared with finite difference and finite element methods.

For spectral and pseudospectral methods, explicit expressions for the expansion coefficients of the derivatives in terms of the expansion coefficients of the solution are needed. Doha [2] obtained a general formula when the basis functions are the ultraspherical polynomials; formulae when the basis functions of the expansion are the first and second kind Chebyshev polynomials and Legendre polynomials are given as a special cases of the ultraspherical polynomials.

El-Hawary [9] introduced a Chebyshev spectral procedure for solving ordinary and partial differential equations by transforming them into integral formulae. He used El-Gendi method (El-Gendi [5]) to obtain an approximation for the finite integrals.

In a recent papers El-Hawary et al. [7] introduce a spectral approximation of integral based on Legendre approximation at the zeros of the first term of the residual. The method is used to solve integral and integro-differential equations. They deduce that the approximation at the zero points is better than approximating it at any other set of points.

El-Hawary et al. [8] derived some useful properties of the ultraspherical polynomials. They introduced an ultraspherical approximation of any continuous function and its finite integrals. They derived error estimation for this approximation. They introduced an algorithm that gives an optimal approximation of the integrals.

Optimal control problems governed by ordinary differential equations are discussed by many authors, among them Egerstedt and Martin [4], Kogan and Eugene [13] and Rampazzo and Sartori [19]. A variety of numerical methods for solving this optimal control problem exists. The most common approach is to replace the unknowns of the problem by some approximation function and to determine the unknowns by minimizing the resulting constrained optimization problem.

Salim [20] introduced a method based on parameterization both the state and control variable. El-Kady [10] uses Chebychev approximation method with El-Gendi matrix to solve the problem.

Martin [16] consider the problem of time-optimal boundary control of a one-dimensional vibrating system subject to a control constraint that prescribes an upper bound for the L^2-norm of the image of the control function under a Volterra operator. He use Newton's method to compute the zero of the optimal value function of certain parametric auxiliary problems, where the steering time is the parameter.

The purpose of this paper is to solve optimal control problem governed by ordinary differential equations. We approximate the highest order derivative in the problem using ultraspherical approximation and obtain approximations to the lower order derivatives by successive integrations.

We introduce two procedures to solve the problems. In the first, we approximate the solution of the problem at specified values of the ultraspherical parameter. The second procedure gives us approximate solution of the problem at the optimal value of the ultraspherical parameter α.

2. Setting of the Problem

We consider the following two optimal control problems:

PROBLEM 1. Find the control u, transferring the system described by:

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = f_i(x_1, x_2, \dots, x_m, u), \quad i = 1, 2, \dots, m,$$
$$0 \leqslant \tau \leqslant T \text{ (specified or free)}, \tag{2.1a}$$

from the position $x_i = x_i(\tau_0)$ to the position $x_i = x_i(\tau_f = T)$ and yielding the minimum of the performance index

$$J = h(x(T), T) + \int_0^T g(x_1, x_2, \dots, x_m, u, \tau)\,\mathrm{d}\tau, \tag{2.1b}$$

with a specified m conditions, where $x = [x_1, x_2, \dots, x_m]$ and $u = [u_1, u_2, \dots, u_{\bar{m}}]$ are state and control variables respectively.

PROBLEM 2. Find the control u, transferring the system described by:

$$x^{(m)}(\tau) = \hat{f}(x, \dot{x}, \dots, x^{(m-1)}, u), 0 \leqslant \tau \leqslant T, \tag{2.2a}$$

from the position $x = x(\tau_0)$ to the position $x = x(\tau_f = T)$ within the time $(\tau_f - \tau_0)$ and yielding the minimum of the performance index:

$$J = h(x(T), T) + \int_0^T g(x, u, \tau)\,\mathrm{d}\tau, \tag{2.2b}$$

with a specified m conditions, where x and u are state and control variables respectively.

The functions f_i, $i = 1, 2, \dots, m$, the scalar functions $\hat{f}$, h and g are generally nonlinear and are assumed to be continuous. The time transformation $\tau = Tt$ is introduced in order to use the ultraspherical integral method defined on the interval $t \in [0, 1]$. It follows that the Eqs. (2.1)–(2.2) are replaced by

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x(t), u(t)), \quad i = 1, 2, \dots, m, \quad 0 \leqslant t \leqslant 1, \tag{2.3a}$$

$$J = h(x(1)) + \int_0^1 g(x, u, t)\,\mathrm{d}t, \tag{2.3b}$$

$$x^{(m)}(t) = \hat{f}(x, \dot{x}, \dots, x^{(m-1)}, u), \quad 0 \leqslant t \leqslant 1 \tag{2.4a}$$

$$J = h(x(1)) + \int_0^1 g(x, u, t)\,\mathrm{d}t. \tag{2.4b}$$

3. Approximation of the System Dynamics and the Performance Index

3.1. SOLUTION OF PROBLEM 1

Let

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \Phi_i(t), \quad i = 1, 2, \ldots, m. \tag{3.1}$$

By integration, and making use of the given conditions, we get

$$x_i(t) = \int_0^t \Phi_i(t)\,\mathrm{d}t + A_i, \quad i = 0, 1, \ldots, m, \tag{3.2}$$

Now we apply our ultraspherical integral approximation, then we have

$$x_i(t_k) = \sum_{j=0}^{N} q_{kj}(\alpha)\Phi_i(t_j) + A_i, \quad i = 1, 2, \ldots, m; k = 0, 1, \ldots, N, \tag{3.3}$$

where $q_{kj}(\alpha), k, j = 0, 1, \ldots, N$ are the elements of the matrix Q as defined by

$$q_{ik}(\alpha) = \sum_{j=0}^{N} (\lambda_j^{[\alpha]})^{-1} \varpi_k^{[\alpha]} C_j^{[\alpha]}(x_k) \int_{-i}^{x_i} C_j^{[\alpha]}(x)\,\mathrm{d}x,$$

where

$$(\varpi_k^{[\alpha]})^{-1} = \sum_{r=0}^{N} (\lambda_r^{[\alpha]})^{-1} (C_r^{[\alpha]}(x_k))^2, x_k \in S$$

and

$$\lambda_j^{[\alpha]} = 2^{j+2\alpha+\tau} j! \frac{\Gamma[\alpha + \frac{1}{2}]\Gamma[j + \alpha + \frac{1}{2}]}{\Gamma[2j + 2\alpha + 1]} \hat{K}_j^{[\alpha]},$$

and

$$\tau = \begin{cases} 1 & \text{if } \alpha = j = 0 \\ 0 & \text{otherwise} \end{cases}.$$

where ,

$$\hat{K}_j^{[\alpha]} = 2^j \frac{\Gamma(j + \alpha)\Gamma(2\alpha + 1)}{\Gamma(j + 2\alpha)\Gamma(\alpha + 1)},$$

$t_k \in S$, $S = \{t_k : C_{N+1}^{[\alpha]}(t_k) = 0, k = 0, 1, \ldots, N\}$, where $C_{N+1}^{[\alpha]}(t)$ is the ultraspherical polynomials defined by Doha [2], and A_i, $i = 0, 1, \ldots, m$ are some unknowns can be defined from the given conditions.

Now, we consider the approximation of the control variables, we shall use the following type of approximation:

$$u_\nu(t) = \sum_{k=0}^{M} a_{\nu k} C_k^{[\alpha]}(t), \tag{3.4}$$

where the unknowns are $\{a_{\nu k}\}$, $\nu = 1, 2, \ldots, \bar{m} : k = 0, 1, \ldots, M$. Making use of these approximations, the optimal control problem (2.3a)–(2.3b) are replaced by the constrained optimization problems

$$\begin{aligned}
\text{Minimize } J = h &\left(\sum_{j=0}^{N} q_{Nj}(\alpha)\Phi_1(t_j) + \right. \\
&\left. + A_1, \sum_{j=0}^{N} q_{Nj}(\alpha)\Phi_2(t_j) + A_2, \ldots, \sum_{j=0}^{N} q_{Nj}(\alpha)\Phi_m(t_j) + A_m \right) + \\
&+ \sum_{i=0}^{N} q_{Ni}(\alpha) g \left(\sum_{j=0}^{N} q_{ij}(\alpha)\Phi_1(t_j) + A_1, \sum_{j=0}^{N} q_{ij}\Phi_2(\alpha)(t_j) + \right. \\
&\left. + A_2, \ldots, \sum_{j=0}^{N} q_{ij}(\alpha)\Phi_m(t_j) + A_m, {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_i), t_i \right),
\end{aligned} \tag{3.4a}$$

subject to

$$\begin{aligned}
I_k = \Phi_i(t_k) - f &\left(\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_1(t_j) + A_1, \sum_{j=0}^{N} q_{kj}(\alpha)\Phi_2(t_j) + \right. \\
&\left. + A_2, \ldots, \sum_{j=0}^{N} q_{kj}(\alpha)\Phi_m(t_j) + A_m, {\sum}' a_{\nu k} C_k^{[\alpha]}(t_k) \right).
\end{aligned} \tag{3.4b}$$

Problem (3.4a)–(3.4b) can be rewritten briefly as:

$$\text{Minimize } J = \tilde{J}(\Phi, \Omega), \text{ subject to } I = \tilde{I}(\Phi, \Omega) = 0, \tag{3.5}$$

where $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_m)$, $\Omega = (a_{\nu 0}, a_{\nu 1}, \ldots, a_{\nu M})$ and $\Phi_k = \Phi_k(x_i), k = 1, 2, \ldots, m; i = 0, 1, \ldots, N$.

Problem (3.5) can be solved at a specified value of the ultraspherical parameter α. For the optimal ultraspherical approximation we construct the following constrained optimization problem instead of (3.5):

$$\text{Minimize } J = \tilde{J}(\Phi, \Omega, \alpha), \text{ subject to } I = \tilde{I}(\Phi, \Omega, \alpha) = 0, \tag{3.7}$$

where α be the ultraspherical parameter.

3.2. SOLUTION OF PROBLEM 2

Let

$$x^{(m)}(t) = \Psi(t), \tag{3.8}$$

where $m \geqslant 1$ and $\Psi(t_i)$, $i = 0, 1, \dots, N$ are some unknowns. By integration, and making use of the given conditions, we get

$$x^{(m-1)}(t) = \int_0^t \Psi(t)\mathrm{d}t + A_0, \tag{3.9a}$$

$$\dots$$

$$x(t) = \underbrace{\int_0^t \int_0^t \cdots \int_0^t \int_0^t}_{m \text{ times}} \Psi(t)\, \mathrm{d}t\, \mathrm{d}t \dots \mathrm{d}t + \sum_{r=0}^{m-1} A_{m-r} t^r. \tag{3.9b}$$

Now we apply our ultraspherical integral approximation, then we have

$$x^{(m-1)}(t_i) = \sum_{j=0}^{N} q_{ij}(\alpha)\Psi(t_j) + A_0, i = 0, 1, \dots, N, \tag{3.10a}$$

$$\dots$$

$$x(t_i) = \sum_{j=0}^{N} a_{ij}^{[m]}(\alpha)\Psi(t_j) + \sum_{r=0}^{m-1} A_r t_i^r, i = 0, 1, \dots, N, \tag{3.10b}$$

and the constants A_r, $r = 0, 1, \dots, m-1$ may be defined from the given conditions. Making use of the same approximation for the control variable, the optimal control problem (2.4a)–(2.4b) are replaced by the constrained optimization problems

$$\text{Minimize } J = h\left(\sum_{j=0}^{N} q_{N_j}^{[m]}(\alpha)\Psi(t_j) + \sum_{r=0}^{m-1} A_r\right) + \sum_{k=0}^{N} q_{Nk}(\alpha) g\left(\sum_{j=0}^{N} q_{kj}^{[m]}(\alpha)\Psi(t_j) + \sum_{r=0}^{m-1} A_r t_k^r, {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k)\right), \tag{3.11a}$$

$$\text{subject to } I_k = \Psi(t_k) - \hat{f}\left(\sum_{j=0}^{N} q_{kj}^{[m]}(\alpha)\Psi(t_j) + \sum_{r=0}^{m-1} A_r t_k^r, \sum_{j=0}^{N} q_{kj}^{[m-1]}(\alpha)\Psi(t_j) + \sum_{r=0}^{m-2} A_r t_k^r, \dots, \sum_{j=0}^{N} q_{kj}(\alpha)\Psi(t_j) + A_0, {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k)\right), \tag{3.11b}$$

Problem (3.11a)–(3.11b) can be rewritten briefly at a specified value of the ultraspherical parameter α and at the optimal value of it as following

$$\text{Minimize } J = \tilde{J}(\Psi, \Omega), \text{ subject to } I = \tilde{I}(\Psi, \Omega) = 0,$$
$$\text{Minimize } J = \tilde{J}(\Psi, \Omega, \alpha), \text{ subject to } I = \tilde{I}(\Psi, \Omega, \alpha) = 0,$$

where $\Psi = (\Psi_0, \Psi_1, \ldots, \Psi_m)$, $\Omega = (a_{\nu 0}, a_{\nu 1}, \ldots, a_{\nu m})$ and $\Psi_k = \Psi(t_k)$.

3.3. PENALTY LEAP FROG METHOD

To solve (3.5) or (3.7), we shall use the approach of penalty function with leap frog algorithm LFOPC [20].

We refer to this method with Penalty leap frog (PLF) method. We develop the PLF method by means of a sequential minimization of the Penalty function

$$P = J + \mu_k \|I\|^2 \tag{3.12}$$

We solve (3.12) as unconstrained optimization problem by means of leap frog algorithm LFOPC algorithm (Snyman [21]). For updating Penalty parameter, we shall use the sequence $\mu_{k+1} = c\mu_k$ with $c \gg 1$, $k = 0, 1, 2, \ldots$.

3.4. ERROR ESTIMATES

THEOREM 3.4.1. *Let $f(t)$ be approximated by ultraspherical polynomials, then there exists a number $\xi = \xi(t)$ in [0,1] such that*

$$f(t) = \sum_{k=0}^{N} a_k C_k^{[\alpha]}(t) + R_N^{[\alpha]}(t, \xi), \tag{3.13a}$$

$$\underbrace{\int_0^{t_i} \int_0^{t_i} \cdots \int_0^{t_i}}_{m-\text{times}} f(t)\,\mathrm{d}t\,\mathrm{d}t \ldots \mathrm{d}t = \sum_{k=0}^{N} q_{ik}^{[m]}(\alpha) f(t_k) + E_N^{[\alpha]}(t_i, \xi) \tag{3.13b}$$

where

$$R_N^{[\alpha]}(t, \xi) = \frac{f^{(N+1)}(\xi)}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} C_{N+1}^{[\alpha]}(t) \tag{3.14a}$$

$$E_N^{[\alpha]}(t_i, \xi) = \frac{f^{(N+1)}(\xi)}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \underbrace{\int_0^{t_i} \int_0^{t_i} \cdots \int_0^{t_i}}_{m-\text{times}} C_{N+1}^{[\alpha]}(t)\,\mathrm{d}t\,\mathrm{d}t \ldots \mathrm{d}t, \tag{3.14b}$$

where

$$\hat{K}_j^{[\alpha]} = 2^j \frac{\Gamma(j+\alpha)\Gamma(2\alpha+1)}{\Gamma(j+2\alpha)\Gamma(\alpha+1)}.$$

Proof. See El-Hawary et al. [7].

THEOREM 3.4.2. *Assume that the optimal control problem (2.3a)–(2.3b) is approximated by our ultraspherical integral method and assuming that* $x_i^{(N+2)}(t)$ *is bounded, i.e.*

$$\|x_i^{(N+2)}(t)\| \leqslant D_i, \text{ then there exists a number } \xi = \xi(t) \text{ in [0,1] such that} \tag{3.15}$$

$$E_{x_i}^{[\alpha]}(t_k, \xi) \leqslant \frac{D_i}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \int_0^{t_k} C_{N+1}^{[\alpha]}(t)\,\mathrm{d}t, \tag{3.16}$$

$$\begin{aligned} E_I^{[\alpha]}(t_k) = {} & f\left(\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_1(t_j) + A_1 + E_N^{[\alpha]}(x_1, \xi),\right. \\ & \sum_{j=0}^{N} q_{kj}(\alpha)\Phi_2(t_j) + A_2 + E_N^{[\alpha]}(x_2, \xi), \dots, \\ & \left.\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_m(t_j) + A_m + E_N^{[\alpha]}(x_m, \xi), {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k) + R_M^{[\alpha]}(t_k, \xi)\right) \\ & - f\left(\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_1(t_j) + A_1, \sum_{j=0}^{N} q_{kj}(\alpha)\Phi_2(t_j) + A_2, \dots,\right. \\ & \left.\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_m(t_j) + A_m, {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k)\right), \end{aligned} \tag{3.17}$$

Proof. Firstly, let $E_{x_i}^{[\alpha]}(t_k, \xi)$ denotes the error in approximation $x_i(t_k)$ with (3.3b), namely

$$E_{x_i}^{[\alpha]}(t_k) = \int_0^{t_k} \Phi_i(t)\,\mathrm{d} - \sum_{j=0}^{N} q_{kj}\Phi_i(t_j), \tag{3.18}$$

then, making use of (3.13)–(3.14), the error in the approximation (3.3b) can be written as:

$$E_{x_i}^{[\alpha]}(t_k, \xi) = \frac{\Phi_i^{(N+1)}(\xi)}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \int_0^{t_k} C_{N+1}^{[\alpha]} t\,\mathrm{d}t = \frac{x^{(N+2)}(\xi)}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \int_0^{t_k} C_{N+1}^{[\alpha]}(t)\,\mathrm{d}t$$

Thus, making use of (3.15),

$$E_{x_i}^{[\alpha]}(t_k, \xi) \leqslant \frac{D_i}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \int_0^{t_k} C_{N+1}^{[\alpha]}(t)\,\mathrm{d}t.$$

Secondly, the original constraint (2.3a) in view of (3.1)–(3.2) becomes

$$I_k = \Phi_i(t_k) - f_i\left(\int_0^{t_k}\Phi_1(t)\,\mathrm{d}t + A_1, \int_0^{t_k}\Phi_2(t)\,\mathrm{d}t + A_2, \dots,\right.$$
$$\left.\int_0^{t_k}\Phi_m(t)\,\mathrm{d}t + A_m, u(t_k)\right) = 0 \tag{3.19}$$

Making use of (3.13), equation (3.19) becomes

$$I_k = \Phi_i(t_k) - f_i\left(\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_1(t_j) + A_1 + E_N^{[\alpha]}(x_1,\xi),\right.$$
$$\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_2(t_j) + A_2 + E_N^{[\alpha]}(x_2,\xi), \dots, .$$
$$\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_m(t_j) + A_m + E_N^{[\alpha]}(x_m,\xi),$$
$$\left.{\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k) + R_M^{[\alpha]}(t_k,\xi)\right).$$

Subtracting (3.4b) from (3.20), we obtain

$$E_I^{[\alpha]}(t_k) = f_i\left(\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_1(t_j) + A_1 + E_N^{[\alpha]}(x_1,\xi),\right.$$
$$\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_2(t_j) + A_2 + E_N^{[\alpha]}(x_2,\xi), \dots,$$
$$\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_m(t_j) + A_m + E_N^{[\alpha]}(x_m,\xi),$$
$$\left.{\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k) + R_M^{[\alpha]}(t_k,\xi)\right)$$
$$- f_i\left(\sum_{j=0}^{N} q_{kj}(\alpha)\Phi(t_j) + A_1, \sum_{j=0}^{N} q_{kj}(\alpha)\phi_2(t_j) + A_2, \dots,\right.$$
$$\left.\sum_{j=0}^{N} q_{kj}(\alpha)\Phi_m(t_j) + A_m, {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k)\right),$$

with $E_N^{[\alpha]}(x,\xi)$ and $R_N^{[\alpha]}(t,\xi)$ are defined in (3.14a) and (3.14b) respectively.

THEOREM 3.4.3. *Assume that the optimal control problem (2.4a)–(2.4b) is approximated by ultraspherical integral method and assuming that* $x^{m+N+1)}(t)$ *is bounded, i.e.*
$\left\| x^{(m+N+1)}(t) \right\| \leqslant D$, then there exists a number $\xi = \xi(t)$ in [0,1] such that (3.21)

$$E_x^{[\alpha]}(t_k, \xi) \leqslant \frac{D}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \underbrace{\int_0^{t_k} \int_0^{t} \cdots \int_0^{t}}_{m-\text{times}} C_{N+1}^{[\alpha]}(t)\,\mathrm{d}t\,\mathrm{d}t \ldots \mathrm{d}t, \tag{3.22}$$

$$\begin{aligned} E_I^{[\alpha]}(t_k) = \hat{f} \Bigg(& \sum_{j=0}^{N} q_{kj}^{[m]}(\alpha)\Psi(t_j) + E_m^{[\alpha]}(x_k, \xi), \\ & + \sum_{r=0}^{m-1} A_r t_k^r, \sum_{j=0}^{N} q_{kj}^{[m-1]}(\alpha)\Psi(t_j) + E_{m-1}^{[\alpha]}(x_k, \xi) \\ & + \sum_{r=0}^{m-2} A_r t_k^r, \ldots, \sum_{j=0}^{N} q_{kj}(\alpha)\Psi(t_j) + E_1^{[\alpha]}(x_k, \xi) \\ & + A_0, {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k) + R_M^{[\alpha]}(t_k, \xi) \Bigg) \\ & - \hat{f} \Bigg(\sum_{j=0}^{N} q_{kj}^{[m]}(\alpha)\Psi(t_j) + \sum_{r=0}^{m-1} A_r t_k^r, \sum_{j=0}^{N} q_{kj}^{[m-1]}(\alpha)\Psi(t_j) \\ & + \sum_{r=0}^{m-2} A_r t_k^r, \ldots, \sum_{j=0}^{N} q_{kj}(\alpha)\Psi(t_j) + A_0, {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k) \Bigg). \end{aligned} \tag{3.23}$$

Proof. Firstly, let $E_x^{[a]}(t_k, \xi)$ denote the error in approximation $x(t_k)$ with (3.10), namely

$$E_x^{[\alpha]}(t_k) = \underbrace{\int_0^{t_k} \int_0^{t} \cdots \int_0^{t}}_{m-\text{times}} \Psi(t)\,\mathrm{d}t\,\mathrm{d}t \ldots \mathrm{d}t - \sum_{j=0}^{N} q_{kj}^{[m]} q_{kj}^{[m]}(\alpha)\Psi(t_j), \tag{3.24}$$

then, making use of (3.13)–(3.14), the error in the approximation (3.10) can be written as:

$$\begin{aligned} E_x^{[\alpha]}(t_k, \xi) &= \frac{\Psi^{(N+1)}(\xi)}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \underbrace{\int_0^{t_k} \int_0^{t} \cdots \int_0^{t}}_{m-\text{times}} C_{N+1}^{[\alpha]}(t)\,\mathrm{d}t\,\mathrm{d}t \ldots \mathrm{d}t \\ &= \frac{x^{(m+N+1)}(\xi)}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \underbrace{\int_0^{t_k} \int_0^{t_k} \cdots \int_0^{t_k}}_{m-\text{times}} C_{N+1}^{[\alpha]}(t)\,\mathrm{d}t\,\mathrm{d}t \ldots \mathrm{d}t \end{aligned}$$

Thus, making use of (3.21),

$$E_x^{[\alpha]}(t_k,\xi) = \frac{D}{(N+1)!\hat{K}_{N+1}^{[\alpha]}} \underbrace{\int_0^{t_k}\int_0^{t}\cdots\int_0^{t}}_{m-\text{times}} C_{N+1}^{[\alpha]}(t)\,\mathrm{d}t\,\mathrm{d}t\ldots\mathrm{d}t.$$

Secondly, the original constraint (2.4a) in view of (3.9) becomes

$$\begin{aligned} I_k = \Psi(t_k) - \hat{f}\Bigg(& \underbrace{\int_0^{t_k}\int_0^{t}\cdots\int_0^{t}}_{m-\text{times}} \Psi(t)\,\mathrm{d}t + \sum_{r=0}^{m-1} A_r t_k^r, \\ & \underbrace{\int_0^{t_k}\int_0^{t}\cdots\int_0^{t}}_{(m-1)-\text{times}} \Psi(t)\,\mathrm{d}t + \sum_{r=0}^{m-2} A_r t_k^r, \ldots, \\ & \int_{-1}^{t_j} \Psi(t)\,\mathrm{d}t + A_0, u(t_k)\Bigg) = 0 \end{aligned} \qquad (3.25)$$

Making use of (3.13), equation (3.25) becomes

$$\begin{aligned} I_k = {} & \Psi(t_k) - \hat{f}\Bigg(\sum_{j=0}^{N} q_{kj}^{[m]}(\alpha)\Psi(t_j) + E_m^{[\alpha]}(x_k,\xi) \\ & + \sum_{r=0}^{m-1} A_r t_k^r, \sum_{j=0}^{N} q_{kj}^{[m-1]}(\alpha)\Psi(t_j) + E_{m-1}^{[\alpha]}(x_k,\xi) \\ & + \sum_{r=0}^{m-2} A_r t_k^r, \ldots, \sum_{j=0}^{N} q_{kj}(\alpha)\Psi(t_j) + E_1^{[\alpha]}(x_k,\xi) \\ & + A_0, {\sum_{k=0}^{M}}' a_{\nu k} C_k^{[\alpha]}(t_k) + R_M^{[\alpha]}(t_k,\xi)\Bigg) = 0. \end{aligned} \qquad (3.26)$$

Subtracting (3.11b) from (3.26), we obtain

$$
\begin{aligned}
E_I^{[\alpha]}(t_k) = \hat{f} \Bigg(& \sum_{j=0}^{N} q_{kj}^{[m]}(\alpha)\Psi(t_j) + E_m^{[\alpha]}(x_k,\xi) \\
& + \sum_{r=0}^{m-1} A_r t_k^r, \sum_{j=0}^{N} q_{kj}^{[m-1]}(\alpha)\Psi(t_j) + E_1^{[\alpha]}(x_k,\xi) \\
& + \sum_{r=0}^{m-2} A_r t_k^r, \ldots, \sum_{j=0}^{N} q_{kj}(\alpha)\Psi(t_j) + E_1^{[\alpha]}(x_k,\xi) \\
& + A_0, \sum_{k=0}^{M}{}' a_{\nu k} C_k^{[\alpha]}(t_k) + R_M^{[\alpha]}(t_k,\xi) \Bigg) \\
& - \hat{f} \Bigg(\sum_{j=0}^{N} q_{kj}^{[m]}(\alpha)\Psi(t_j) + \sum_{r=0}^{m-1} A_r t_k^r, \sum_{j=0}^{N} q_{kj}^{[m-1]}(\alpha)\Psi(t_j) \\
& + \sum_{r=0}^{m-2} A_r t_k^r, \ldots, \sum_{j=0}^{N} q_{kj}(\alpha)\Psi(t_j) + A_0, \sum_{k=0}^{M}{}' a_{\nu k} C_k^{[\alpha]}(t_k) \Bigg).
\end{aligned}
$$

with $E_N^{[\alpha]}(x,\xi)$ and $R_N^{[\alpha]}(t,\xi)$ are defined in (3.14a) and (3.14b) respectively.

REMARKS

(1) The error estimate for the dynamic system can be taken as:

$$EDS = \sum_{k=0}^{N} |E_I^{[\alpha]}(t_k)|^2,$$

this value can be measured by the maximum error of the constraints (3.15b) or (3.15c). The maximum error at the boundary conditions MEBC are used as an index of computational efficiency and accuracy of the ultraspherical integral method (see Elnagar [10]).

(2) To satisfy the constraint $\alpha > -\frac{1}{2}$, we make the following change of variable:

$$\alpha = e^{(s^2+\varepsilon)} - \frac{3}{2}, 0 < \varepsilon \ll 1,$$

and then the problem (3.18) depends on s which has no constraints.

4. Numerical Examples

Now, we consider the following problems to show the effectiveness of our technique.

EXAMPLE 4.1. (The Feldbaum Problem).
The object is to find the optimal control $u(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]\,\mathrm{d}t, \text{ subject to } \frac{\mathrm{d}x}{\mathrm{d}t} = -x + u, 0 \leqslant t \leqslant 1$$
$$\text{with } x(0) = 1,$$

this problem was considered by Van Dooren and Vlassenbroeck [23]. We give the approximation (3.1)–(3.4) for the state variable so

$$x(t_i) = \sum_{k=0}^{N} q_{ik}\Phi_k + 1.$$

We also use the approximation (3.4) for the control variable, then problem can be converted into the following constrained optimization problem:

$$\text{Minimize } I = \frac{1}{2}\sum_{i=0}^{N} q_{N_i}\left[\left[\sum_{k=0}^{N} q_{ik}\Phi_k + 1\right]^2 + \left[{\sum_{k=0}^{M}}' a_k C_k^{[\alpha]}(t_i)\right]^2\right],$$

Subject to

$$g_i = \Phi(t_i) + \left[\sum_{k=0}^{N} q_{ik}\Phi_k + 1\right] - {\sum_{k=0}^{M}}' a_k C_k^{[\alpha]}(t_i), \quad i = 0, \ldots, N$$

Using our PLF technique, we can solve this constrained optimization problem. Table 1 presents the optimal cost J^* and maximum absolute error EDS in the constraints at optimal value of α and at some special values of it. In Table 2 we present a comparison between the ultraspherical integral solution and those obtained by Jacobson et al. [14] making use of Classical Chebyshev method, El-Gindy [5] making use of Chebyshev Spectral method and Elnagar [10] making use of Cell Averaging Chebyshev method.

Table I. Results of Example 4.1, $N = M = 4$

α	J^*	EDS
0.0	0.19290901	6.34E-15
0.5	0.1929093	8.02E-15
1.0	0.192909736	3.08E-14
$\alpha^* = 0.421$	0.192909281	1.15E-14

Table II. Comparison of ultraspherical solution and ref [14]

Method	J^*
Classical Chebyshev [14]	
$M = 7, N = 10$	0.1929030
$M = 9, N = 15$	0.192909298
Cell averaging [10]	
$M = N = 5$	0.192909288
$M = N = 7$	0.192909298
Chebyshev Spectral [5]	
$M = 5, N = 7$	0.192909292
$M = 7, N = 9$	0.192909299
$M = 9, N = 11$	0.192909298
Present method, $M = N = 4$	0.192909281
Exact Solution	0.192909281

EXAMPLE 4.2. Among all piecewise differentiable control variables, find the optimal control $u(t)$ which minimizes

$$I = \int_0^1 [x_1^2(t) + x_2^2(t) + 0.005u^2(t)]\,\mathrm{d}t,$$

subject to $\ddot{x}(t) + \dot{x}(t) - u(t) = 0$, $\dot{x}(0) = -1$ and $x_1(t) - 8(t - 0.5)^2 + 0.5 \leqslant 0$. We give the approximation (3.1)–(3.4) for the state variable so

$$x(t_i) = \sum_{k=0}^{N} q_{ik}^{[2]} \Phi_k - t_i, \quad \dot{x}(t_i) = \sum_{k=0}^{N} q_{ik} \Phi_k - 1.$$

We also use one of the approximation (3.4) for the control variable, then the problem can be converted to the following constrained optimization problem:

$$\text{Minimize } I = \frac{1}{2} \sum_{i=0}^{N} q_{N_i} \left[\left[\sum_{k=0}^{N} q_{ik}^{[2]} \Phi_k - t_i \right]^2 + \left[\sum_{k=0}^{N} q_{ik} \Phi_k - 1 \right]^2 + 0.005 \left[{\sum_{k=0}^{M}}' a_k C_k^{[\alpha]}(t_i) \right]^2 \right],$$

Subject to

$$g_i = \Phi(t_i) + \left[\sum_{k=0}^{N} q_{ik} \Phi_k - 1 \right] - {\sum_{k=0}^{M}}' a_k C_k^{[\alpha]}(t_i) = 0,$$

Table III. Results of example 4.2, $N = M = 4$

α	J^*	EDS
0.0	0.766685	1.25E-07
0.5	0.75092	2.29E-07
1.0	0.739867	2.21E-07
α^*=0.986	0.730694	2.19E-07

Table IV. Comparison of ultraspherical solution and Ref. [10]

Method	J^*
Cell Averaging [10]	
$M = N = 9$	0.74096103
Present method, $M = N = 4$	0.7306941

We approximate the inequality constraint by adding a slack variable as we show previously.

$$g_i\left(\sum_{k=0}^{N} q_{ik}^{[2]}\Phi_k - t_i\right) - 8(t_i - 0.5)^2 + 0.5 + \Phi_{N+i}^2 = 0$$

Table 3 presents the optimal cost J^* and maximum absolute error EDS in the constraints at optimal value of α and at some special values of it. In Table 4 we present a comparison between the ultraspherical integral solution and those obtained by Elnagar [10] making use of Cell Averaging Chebyshev method.

EXAMPLE 4.3. (The minimum time orbit transfer problem)
One of the best known trajectory optimization examples is the problem of minimizing the transfer time of a constant low-thrust ion rocket between the orbits of Earth and Mars. This involved the determination of the thrust angle history, for which no exact solution is known [5]. The performance index of the problem can be stated as follows: minimize $I = T$, subject to the following time-varying equations:

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = x_2, \; \frac{\mathrm{d}x_2}{\mathrm{d}\tau} = \frac{x_3^2}{x_1} - \frac{\gamma}{x_1^2} + \frac{R_0 \sin u}{m_0 + m\tau},$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}\tau} = -\frac{x_2 x_2}{x_1} + \frac{R_0 \cos u}{m_0 + m\tau}, 0 \leqslant \tau \leqslant T, \text{ with the boundary conditions}$$
$$x_1(0) = 1.0, \quad x_2(0) = 0.0, \quad x_3(0) = 1.0$$
$$x_1(T) = 1.525, \quad x_2(T) = 0.0, \quad x_3(T) = 0.8098,$$

where T is the unknown final time to be minimized, γ characterizes the gravitational attraction from the sum, R_0 is the constant low-thrust magnitude, u is the control angle measured from the local horizontal, m_0 is the initial mass, m is the constant propellant consumption rate.

Using normalized values [23], we have $\gamma = 1, R_0 = 0.1405, m_0 = 1$ and $m = -0.07487$. Transforming the domain $\tau \in [0, T]$ to $t \in [0, 1]$, we have: minimize $J = T$, subject to the following time-varying equations:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = Tx_2, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = T\left[\frac{x_3^2}{x_1} - \frac{\gamma}{x_1^2} + \frac{R_0 \sin u}{m_0 + mTt}\right],$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = T\left[-\frac{x_2 x_3}{x_1} + \frac{R_0 \cos u}{m_0 + mTt}\right], 0 \leqslant t \leqslant 1, \text{ with the boundary conditions}$$

$$x_1(0) = 1.0, \quad x_2(0) = 0.0, \quad x_3(0) = 1.0$$

$$x_1(1) = 1.525, \quad x_2(1) = 0.0, \quad x_3(1) = 0.8098,$$

Since the state variables x_1, x_2, x_3 are defined in the two boundaries, we shall use the ultraspherical approximation of the second derivatives, namely,

$$x_1''(t) = \phi(t), x``_2(t) = \psi(t), x_3''(t) = \theta(t),$$

By integration and using boundary conditions and making use of the ultraspherical integral approximation (3.1)–(3.4), so

$$x_1'(t_i) = \sum_{j=0}^{N}(q_{ij} - q_{Nj}^{[2]})\phi_j + 0.525,$$

$$x_2'(t_i) = \sum_{j=0}^{N}(q_{ij} - q_{Nj}^{[2]})\psi_j,$$

$$x_3'(t_i) = \sum_{j=0}^{N}(q_{ij} - q_{Nj}^{[2]})\theta_j - 0.1908$$

$$x_1(t_i) = \sum_{j=0}^{N}(q_{ij}^{[2]} - t_i q_{Nj}^{[2]})\phi_j + 0.525t_i,$$

$$x_2(t_i) = \sum_{j=0}^{N}(q_{ij}^{[2]} - t_i q_{Nj}^{[2]})\psi_j,$$

$$x_3(t_i) = \sum_{j=0}^{N}(q_{ij}^{[2]} - t_i q_{Nj}^{[2]})\theta_j - 0.1908$$

We also use the approximation (3.4) for the control variable, then the problem can be converted to the constrained optimization problem (3.5), which can be solved using our PLF technique.

Table V. Results of example 4.3, $N = M = 5$

α	J^*	EDS
0.0	3.315517	6.35E-06
0.5	3.315662	7.21E-06
1.0	3.316819	1.26E-06
$\alpha^* = 0.519$	3.311212	8.77E-07

Table VI. Comparison of ultraspherical solution and Ref. [5]

Method	T	MEBC
Moyer and Pinkham [17]		
Gradient		
First	3.317	0.10%
Second	3.17	0.05%
Falb and de Jong [11]	3.3193	
Hontoir and Cruz [13]	3.3194	
Taylor and Constantinides[22]	3.3819	0
El-Gindy et al. [5]	3.3117	0
Present method	3.31121	0

Table 5 presents the optimal cost J^* and maximum absolute error EDS in the constraints at optimal value of α and at some special values of it. In Table 6 we present a comparison between the ultraspherical integral solution and those obtained by [5] making use of Chebyshev Spectral method.

EXAMPLE 4.4. (Controlled duffing oscillator)
Recently, special attention has been devoted to the study of the controlled duffing oscillator which is known to describe many important oscillating phenomena in nonlinear engineering systems.

THE CONTROLLED LINEAR OSCILLATOR

The optimal control of a linear oscillator governed by the differential equation can be considered as:

$$\ddot{x} + \omega^2 x = u, \text{ with } x(-T) = x_{10}, x(-T) = x_{20}, -T < \tau < 0,$$
$$\dot{x}(0) = 0, \quad \dot{x}(0) = 0,$$

in which a dot means differentiation with respect to τ and T is specified.

One wish to control the state of this plant such that the performance index

$$I = \frac{1}{2}\int_{-T}^{0} u^2(\tau)\,\mathrm{d}\tau,$$

is minimized over all admissible control function $u(\tau)$.

The time transformation $\tau = T(t-1)$ is introduced in order to use the ultraspherical polynomials on the interval [0,1]. The optimal control problem then be represented as follows:

$$\text{Minimize } I = \frac{T}{2}\int_{0}^{1} u^2(t)\,\mathrm{d}t,\ \text{ subject to } x'' + T^2(\omega^2 x - u) = 0, \quad 0 < t < 1,$$

in which a dash means differentiation with respect to t,

$$x(0) = x_{10}, x'(0) = x_{20}, \quad x(1) = 0, \quad x'(1) = 0.$$

To solve this system, we put $x^{(4)}(t) = \phi(t)$ and making use of ultraspherical integral approximation we can approximate the state variable and its derivatives as follows:

$$x^{(r)}(t_i) = \sum_{j=0}^{N} \ell_{ij}^{(r)}\phi(t_j) + \mathrm{d}_i^{(r)},\ \text{ where } x^{(0)}(t_i) = x(t_i),$$

$$\begin{aligned}
\ell_{ij}^{(3)} &= q_{ij} + 12q_{Nj}^{[4]} - 6q_{Nj}^{[3]}, \ell_{ij}^{(2)} = q_{ij}^{[2]} + t(12q_{Nj}^{[4]} - 6q_{Nj}^{[3]}) + 2q_{Nj}^{[3]} - 6q_{Nj}^{[4]},\\
\ell_{ij}^{(1)} &= q_{ij}^{[3]} + \frac{1}{2}t^2(12q_{Nj}^{[4]} - 6q_{Nj}^{[3]}) + t(2q_{Nj}^{[3]} - 6q_{Nj}^{[4]}),\\
\ell_{ij}^{(0)} &= q_{ij}^{[4]} + \frac{1}{6}t^3(12q_{Nj}^{[4]} - 6q_{Nj}^{[3]}) + \frac{1}{2}t^2(2q_{Nj}^{[3]} - 6q_{Nj}^{[4]}),\\
d_i^{(3)} &= 6Tx_{20} + 12x_{10}, d_i^{(2)} = t(6Tx)_{20} + 12x_{10}) - 4Tx_{20} - 6x_{10},\\
d_i^{(1)} &= \frac{1}{2}t^2(6Tx_{20} + 12x_{10}) + t - (4Tx_{20} + 6x_{10}) + Tx_{20},\\
d_i^{(0)} &= \frac{1}{6}t^3(6Tx_{20} + 12x_{10}) + \frac{1}{2}t^2(4Tx_{20} + 6x_{10}) + tTx_{20} + x_{10}.
\end{aligned}$$

We also use the approximation (3.4) for the control variable, then the problem can be converted to the constrained optimization problem (3.5), which can be solved using our PLF technique.

Table 7 presents the optimal cost J^* and maximum absolute error EDS in the constraints at optimal value of α and at some special values of it. In Table 8 we present a comparison between the ultraspherical integral solution and those obtained by El-Gindy [5] making use of Chebyshev spectral method.

Table VII. Results of example 4.4, $N = M = 8$

α	J^*	EDS
0.0	0.185184	3.06E-06
0.5	0.185934	3.39E-06
1.0	0.189413	6.83E-06
$\alpha^* = 0.3573$	0.1848509	2.61E-06

Table VIII. Comparison of ultraspherical solution and ref. [5]

Method	J^*
El-Gindy et al.[5]	
$N = 15$, $M = 7$	0.1848512
Present method	0.1848509

THE CONTROLLED DUFFING OSCILLATOR

Let us now investigate the optimal control of the duffing oscillator described by the nonlinear differential equation:

$$\ddot{x} + \omega^2 x + \varepsilon x^3 = u,$$

subject to the same boundary conditions and performance index as before.

The approach system dynamics, boundary conditions, and performance index take the same expression. Table 9 presents the optimal cost J^* and maximum absolute error EDS in the constraints at optimal value of α and at some special values of it. In Table 10 we present a comparison between the ultraspherical integral solution and those obtained by El-Gindy [5] making use of Chebyshev spectral method.

Table IX. Results of Example 4.5, $N = M = 8$

α	J^*	EDS
0.0	0.1851838	3.06E-06
0.5	0.1889957	4.02E-06
1.0	0.1927229	1.63E-06
$\alpha^* = 0.305$	0.1874189	2.65E-06

Table X. Comparison of ultraspherical solution and Ref. [5]

Method	J^*
El-Gindy et al.[5]	
$N = 15, M = 7$	0.187433791
Present method	0.187418921

5. Conclusion

The basic idea of our present method is to transform the optimal control problems governed by ordinary differential equations to a constrained optimization problem, by using ultraspherical approximation method. We solve the resulting constrained optimization problem since it is easier than solving the original problem. Here we use PLF method, which may be more suitable in such case, where the number of constraints is increases.

The major advantages of this method is that, we can deal directly with the highest-order derivatives in the differential equation without transforming it to a system of first order, and that will reduce the number of the unknowns. In this way, the optimal control problem is replaced by a parameter optimization problem which consists of the minimization of the performance index subject to algebraic constraints. Finally, the method has been extended to the linear and nonlinear optimal control problems.

The tables given previously show that the suggested technique is quite reliable. It can be successfully applied to both linear and nonlinear ordinary differential problems and related optimal control problems. The methods produce an accurate solution at small number of nodes. The comparison of the maximum absolute error resulting from the proposed method and those obtained by El-Gindy et al. [5], Elnagar [10] and Jacobson et al. [14] show favorable agreement and always it is more accurate than these treatments.

References

1. Canuto, C., Hussaini, M. Y., Quarteroni, A. and Zang, T. A. (1988), *Spectral Methods in Fluid Dynamic*, Springer, New York.
2. Doha, E. H. (1998), The ultraspherical coefficients of moments of a general order derivatives of an infinitely differentiable function *J. Comput. and Appl. Math.* 89: 53–72.
3. Egerstedt, M. and Martin, C. (1998), A control theoretic model of muscular actions in human head-eye coordination, *J. of Mathematical System, Estimation and Control* 8: 1–19.
4. EL-Gendi, S.E. (1969), Chebyshev solutions of differential, integral, and integro-differential equations, *Computer J.* 12: 282–287.
5. El-Gindy, T. M., El-Hawary, H. M., Salim, M. S. and El-Kady, M. (1995), A Chebyshev approximation for solving optimal control problems, *Computers Math. Appl.* 29(6): 35-45.

6. El-Hawary, H.M., Salim, M.S and Hussien, H.S. (2000), Legendre spectral method for solving integral and integro-differential equations *Int. J. of Computer Math.* 75: 187–203.
7. El-Hawary, H. M., Salim, M. S and Hussien, H. S. (2000), An optimal ultraspherical approximation of integrals, *Int. J. of Computer Math.* 77(1/2): (to appear).
8. El-Hawary, H. M. (1990), Numerical treatment of differential equations by spectral methods, Ph. D. Dissertation, Assiut University.
9. El-Kady, M. M. (1994), Numerical studies for optimal control problems, Ph.D. Thesis, Assuit University.
10. Elnagar, G. M. (1997), State-control spectral Chebyshev parameterization for linearly constrained quadratic optimal control problems, *J. Comput. and Applied Math.* 97: 19–40.
11. Falb, P.L. and de Jong, J. L. (1969), Some successive approximation methods in control and oscillation theory, Academic Press, New York.
12. Gottlieb, D. and Orszag, S.A. (1977), Numerical analysis of spectral methods: theory and application, *CBMS-NSF Regional Conference Series in Applied Mathematics*, 26: SIAM, Philadelphia.
13. Hontoir, Y. and Cruz, J. B. (1972), A manifold imbedding algorithm for optimization problems, *Automatica* 8: 581–588.
14. Jacobson, D. H. and Lele, M. M. (1969), *Iee Trans. Automat. Control* A C-14.
15. Kogan, K. and Eugene, K. (1998), Tracking demands in optimal control of managerial systems with continuously-divisible , double constrained resources, *J. Global Optimization*, 13: 43–59.
16. Martin, G. (2000), A Newton method for the computation of time-optimal boundary controls of one-dimensional vibrating systems, Journal of Comput. and Applied Math. 114: 103–119.
17. Moyer, H. G. and Pinkham, G. (1964), Several trajectory optimization techniques: Part II: Application. In: Balakrishnan, A. V. and Neustadt, L. W., (eds). Computing Methods in Optimization Problems, Academic Press, New York 91–109.
18. Rampazzo, F. and Sartori, C. (1998), The minimum time function with unbounded control, Egerstedt and Martin, *J. of Mathematical System, Estimation and Control*, 8(2): 1-34.
19. Salim, M. S. (1990), Numerical studied of optimal control problems and its applications, Ph.D. Thesis, Assiut university, Egypt.
20. Snyman, J.A. (1989), A convergent dynamic method for large minimization problems, *Computers Math. Appl.* 17(10): 1369–1377.
21. Szegö, (1985), Orthogonal Polynomials, *Am. Math. Soc. Colloq. Pub.* 23.
22. Taylor, J. M. and Constantinides, C. T. (1972), Optimization: application of the epsilon method, *IEEE Trans. Automat. Contr. Ac.* 17: 128–131.
23. Van Dooren, R. and Vlassenbroeck, J. (1988), A Chebyshev technique for solving nonlinear optimal control problems, *IEEE Trans. Automat. Contr.*, 33 (4): 333–339.